Hyperbolicity of linear systems on abelian varieties

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Introduction

All varieties are defined over $\mathbb{C}$. Before focusing on abelian varieties, recall two facts:

We have effective results about the algebraic hyperbolicity of a very general hypersurface in $\mathbb{P}^n$ (Clemens, Ein, Voisin, ...)

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Hypersurface in $\mathbb{P}^n$: zero locus of a hom. polynomial of deg d, $\in |\mathcal{O}_{\mathbb{P}^n}(d)|$.

By "effective", I mean an effective estimate on d such that the zero locus is algebraically hyperbolic.

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It is often useful to consider adjoint bundles:

$$K_X + mL\,.$$

Introduction

From these two facts:

Conjecture (Moraga-Yeong, '24)

Let L be an ample line bundle on a smooth projective variety X of dimension $n \geq 2$. If

$$m \geq 3n + 1\,,$$

then the adjoint linear system $|K_X + mL|$ is algebraically hyperbolic, i.e., a very general element in it is algebraically hyperbolic.

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The bound $3n + 1$ is optimal: curves in $\mathbb{P}^2$ ($X = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(1)$).

They proved the conjecture when X is a toric variety.

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Y is Brody hyperbolic, if there exist no non-constant entire curves $\mathbb{C} \to Y$. In particular, Y does not contain rational or elliptic curves.

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Y is Kobayashi hyperbolic $\overset{\text{Demailly}}{\Longrightarrow}$ Y is algebraically hyperbolic.

Y is algebraically hyperbolic, if $\exists\, \epsilon > 0$ s.t. $\forall$ smooth projective curve C and $\forall$ non-constant morphism

$$f\colon C \to Y$$

one has

$$2g(C) - 2 \geq \epsilon \cdot \deg_C(f^*L|_Y)\,.$$

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Demailly conjectured that the converse implication holds true in general. In our situation, this is true:

Y is alg. hyperbolic $\Longrightarrow \mathrm{Sp}(Y) = \emptyset \overset{\text{Bloch}}{\Longrightarrow}$ Y is Brody hyperbolic,

where $\mathrm{Sp}(Y)$ is the union of translates of positive dimensional abelian subvarieties of A, which are contained in Y.

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Theorem (C., '25)

(A, L) : polarized abelian variety of dimension $g \geq 3$.
Then, $|mL|$ is hyperbolic if

$$m \geq g - 1\,,$$

unless

$$(A, L) \simeq (E, \Theta) \times (A', L')$$

where (E, Θ) is a principally polarized elliptic curve.
In this exceptional case, one needs $m \geq g$.

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Ingredients of the proof:

1) A recent hyperbolicity criterion of Ito-Moraga-Raychaudhury-Yeong (IMRY);

2) Recent results about sheaves on abelian varieties due to Pareschi and Alvarado-Pareschi;

3) The *basepoint-freeness threshold* of (A, L).

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3) The *basepoint-freeness threshold* of (A, L).

I will just explain the proof of a weaker result, which is already due to IMRY.

The bpf threshold of (A, L)

Notation: $\mathcal{F}$ a coherent sheaf on A, $t \in \mathbb{Q}$.
The $\mathbb{Q}$-*twisted sheaf* $\mathcal{F}\langle tL \rangle$ is a formal symbol for the pair $(\mathcal{F}, tL)$.

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Over A, it is possible to define the *cohomological ranks* of such objects (Jiang-Pareschi):

$$h^i(A, \mathcal{F}\langle tL\rangle) := \frac{1}{b^{2g}} \cdot h^i(A, \mu_b^*(\mathcal{F}) \otimes L^{ab} \otimes \alpha)\,,$$

where $t = \frac{a}{b}$, with $b > 0$, $\alpha \in \mathrm{Pic}^0 A$ is general, and

$$\mu_b \colon A \to A, \quad p \mapsto bp\,.$$

The definition works as: $\mu_b^* L \sim_{\mathrm{alg}} L^{b^2}$, and so $L^{ab} =$ "$\mu_b^*(\frac{a}{b} L)$", and $b^{2g} = \deg \mu_b$.

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gives us a function:

$$h^i_{\mathcal{F},L} \colon \mathbb{Q} \to \mathbb{Q}_{\geq 0}, \quad t \mapsto h^i_{\mathcal{F},L}(t) = h^i(A, \mathcal{F}\langle tL\rangle)$$

Definition (bpf threshold)

Let $\mathcal{I}_0 \subseteq \mathcal{O}_A$ be the ideal sheaf of $0 \in A$, then

$$\beta(A, L) := \mathrm{Inf}\left\{t \in \mathbb{Q} \mid h^1_{\mathcal{I}_0,L}(t) = 0\right\}.$$

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- $\beta(A, L) \in (0, 1]$, and it is < 1 iff L is basepoint-free;
- $\beta(A, mL) = \frac{\beta(A,L)}{m}$.
- E.g., $\beta(A, m\Theta) = \frac{1}{m}$.

What is the relation with hyperbolicity of $|L|$?

Theorem (IMRY)

If $\beta(A, L) < \frac{1}{g-1}$, then $|L|$ is hyperbolic.

As a corollary, we get that $|mL|$ is hyperbolic if $m \geq g$.

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Indeed:

$$\beta(A, mL) = \frac{\beta(A, L)}{m} \leq \frac{1}{m} < \frac{1}{g-1}\,.$$

In order to prove the Theorem, one applies a more general hyperbolicity criterion of IMRY.

$\beta(A, L) < \frac{1}{g-1} \Rightarrow |L|$ is hyperbolic: Proof

Criterion (IMRY)

Let X be a smooth proj. variety of dim. $n \geq 3$, with T_X nef. Let N (resp. L) be a globally generated (resp. ample) line bundle on X. Assume $\exists\, \delta$ a positive rational number s.t.:

i) $M_N \langle \delta L \rangle$ is a nef $\mathbb{Q}$-twisted bundle, where

$$0 \to M_N \to H^0(X, N) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} N \to 0$$

ii) $K_X + N - \delta(n-2)L$ is ample.

Then, $|N|$ is algebraically hyperbolic.

Here, "nef $\mathbb{Q}$-twisted bundle" means that $\mathcal{O}_{\mathbb{P}(M_N)}(1) + \delta\pi^* L$ is a nef $\mathbb{Q}$-divisor, where $\pi \colon \mathbb{P}(M_N) \to X$.

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Let us apply it when $X = A$ and $N = L$.

$\beta(A,L) < \frac{1}{g-1} \Rightarrow |L|$ is hyperbolic: Proof

Going back to (A, L):

If $\beta(A,L) < r_0 < 1$, then L is globally generated *and*

$$M_L \left\langle \frac{r_0}{r_0 - 1} L \right\rangle$$

is ample.
This is a computation, using the Fourier-Mukai-Poncaré transform, due to Jiang-Pareschi + a result of Debarre.

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If $r_0 = \frac{1}{g-1}$, then $M_L \left\langle \frac{1}{g-2} L \right\rangle$ is ample. Take $\delta < \frac{1}{g-2}$ s.t. $M_L \langle \delta L \rangle$ is still ample (hence nef). Since

$$L - \delta(g-2)L = \underbrace{(1 - \delta(g-2))}_{>0} L$$

is ample, the Criterion applies to $|L|$.

More about $\beta(A, L)$

$\beta(A, L)$ is also related to other positivity properties of (A, L):
- syzygies;
- higher order embeddings;
- ...

This suggested a *Fujita-type* upper bound for the bpf threshold:

Conjecture 1 (C.)

$$\beta(A, L) \leq \mathrm{Max}_B \left\{ \frac{\dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\},$$

where $B \subseteq A$ varies among all non-zero abelian subvarieties of A.

Conjecture 1 (C.)

$$\beta(A, L) \leq \mathrm{Max}_{\{0\} \neq B \subseteq A} \left\{ \frac{\dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\}$$

Why of "Fujita-type"?

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$$\beta(A, L) \leq \mathrm{Max}_{\{0\} \neq B \subseteq A} \left\{ \frac{\dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\}$$

Why of “Fujita-type”? If we impose that the RHS is < 1, we get:

Corollary (assuming Conjecture 1)

If $(L^{\dim B} \cdot B) > (\dim B)^{\dim B}$ for all $\{0\} \neq B \subseteq A$, then L is basepoint-free.

We have similar implications for the very ampleness of L, and even for the higher syzygies of L.

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$$\beta(A, L) \leq \mathrm{Max}_{\{0\} \neq B \subseteq A} \left\{ \frac{\dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\}$$

Similarly, if we impose that the RHS is $< \frac{1}{g-1}$, we get:

Conjecture 2 (C.)

If $(L^{\dim B} \cdot B) > ((g-1) \dim B)^{\dim B}$ for all $\{0\} \neq B \subseteq A$, then the linear system $|L|$ is hyperbolic.

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What is known: Conj. 1 (and hence Conj. 2) is true if $g = 2$ (C.), if $g = 3$ (Ito), "essentially true" when $g = 4$ (Jiang).
In arbitrary dimension, it is known "up to a factor 2" (Jiang).

Theorem (Jiang)

$$\beta(A, L) \leq \mathrm{Max}_{\{0\} \neq B \subseteq A} \left\{ \frac{2 \cdot \dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\}$$

Corollary

If $(L^{\dim B} \cdot B) > (2 \cdot (g-1) \dim B)^{\dim B}$ *for all* $\{0\} \neq B \subseteq A$*, then* $|L|$ *is hyperbolic.*

Birational geometry bounds $\beta(A, L)$

Take a rational number $t \in \mathbb{Q}$ s.t.

$$\mathrm{Max}_{\{0\} \neq B \subseteq A} \left\{ \frac{\dim B}{\sqrt[\dim B]{(L^{\dim B} \cdot B)}} \right\} < t \overset{??}{\Longrightarrow} \beta(A, L) < t\,.$$

In particular,

$$((tL)^g) > g^g$$

$\rightsquigarrow$ $\exists$ an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} tL$ such that $0 \in \mathrm{Nklt}(A, D)$.

Proposition (Lozovanu, Ito)

If $\mathrm{Nklt}(A, D)$ contains an abelian subvariety of A (possibly $\{0\}$) as an irreducible component, then $\beta(A, L) < t$.

Birational geometry bounds $\beta(A, L)$

If $\mathrm{Nklt}(A, D)$ contains an abelian subvariety of A as an irreducible component, then $\beta(A, L) < t$.

Ito and Jiang applied the Angehrn-Siu-Helmke's method to abelian varieties: to cut the non-klt locus.

$g = 3$. Take the minimal lc center Z of (A, D) at 0. Up to tie-breaking, we may assume that Z is the only irreducible component of $\mathrm{Nklt}(A, D)$ passing through 0. Either Z is an abelian subvariety of A, or it is possible to cut it, that is we may slightly perturb $D \rightsquigarrow D'$ s.t. $D' \sim_{\mathbb{Q}} t'L$, with $0 < t - t' \ll 1$, and the minimal lc center Z' of (A, D') at 0 is *strictly* contained in Z.

Further remarks

Remark (C.)

If $g = 2$ and (A, L) is a polarized abelian surface, then $|L|$ is hyperbolic as soon as L is basepoint-free.
In particular, $|mL|$ is hyperbolic if $m \geq 2$ and L is just ample.

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Proof: Take $D \in |L|$ general. Then, D is a smooth proj. curve $\subseteq A$. Take

$$0 \to \mathcal{O}_A(-D) \to \mathcal{O}_A \to \mathcal{O}_D \to 0$$

and apply the holom. Euler characteristic:

$$0 = \chi(\mathcal{O}_A) = \chi(\mathcal{O}_A(-D)) + \chi(\mathcal{O}_D) = h^2(\mathcal{O}_A(-D)) + (1 - g(D))\,.$$

Therefore,

$$g(D) = 1 + h^0(\mathcal{O}_A(D)) > 1\,.$$

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If $g = 2$ and (A, L) is a polarized abelian surface, then $|L|$ is hyperbolic as soon as L is basepoint-free.

Theorem (IMRY, 2nd ArXiv version)

If $g \geq 3$ and L separates $(g-2)$-jets, then $|L|$ is hyperbolic.

Here, "L separates $(g-2)$-jets" means

$$H^0(A, L) \twoheadrightarrow H^0(A, L \otimes \mathcal{O}_A/\mathcal{I}_p^{g-1}) \quad \forall p \in A\,.$$

- 0-jet separation $\Leftrightarrow$ basepoint-freeness.
- They also gave an alternative proof of our main theorem along these lines.

Further remarks

If A is a *simple* abelian variety, Conj. 1 reduces to ask

$$\beta(A,L) \leq \frac{g}{\sqrt[g]{(L^g)}} .$$

Jiang proved this inequality for a *very general* abelian variety of dim. $g \leq 6$, but for $g \geq 7$ it is still open!

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If A is simple, then any $|L|$ is hyperbolic, simply because the special locus of any closed subvariety $\subsetneq A$ is empty.

Anyway, Conj. 1 for simple abelian varieties would lead to interesting consequences regarding hyperbolicity!

Further remarks

It suggests the weaker

Conjecture 2'

Let (A, L) be a general polarized abelian variety of dim. g and type $(d_1, \dots, d_g)$. If

$$h^0(A, L) > \frac{(g(g-1))^g}{g!},$$

then $|L|$ is hyperbolic.

Proof (assuming Conj. 1 for simple / very general abelian varieties): Take (A_0, L_0) a simple ab. variety in the same moduli space. Then,

$$\beta(A_0, L_0) \leq \frac{g}{\sqrt[g]{(L_0^g)}} = \frac{g}{\sqrt[g]{g! \cdot h^0(A, L)}} < \frac{1}{g-1}.$$

Since the bpf threshold is upper-semicontinuous, we get $\beta(A, L) < \frac{1}{g-1}$ for a general (A, L).

Further remarks

One has

$$\frac{1}{\sqrt[g]{h^0(A,L)}} \leq \beta(A,L) \stackrel{?}{\leq} \frac{g}{\sqrt[g]{(L^g)}} = \frac{g}{\sqrt[g]{g!}} \cdot \frac{1}{\sqrt[g]{h^0(A,L)}}$$

for a very general (A, L) (the first inequality indeed holds true for any (A, L) by Ito).

It is expected that $\beta(A,L) \sim \frac{1}{\sqrt[g]{h^0(A,L)}}$ when (A, L) is general, but there isn't a precise conjecture: even better estimates than the one in Conj. 2' may be possible in some cases.

THANK YOU!

GRAZIE!